Measures and combinatorics on λ^+

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- In fact (assuming large cardinals), in $L(\mathbb{R})$, $\omega_1 \to (\omega_1)^{\omega_1}$ (Martin).
- Question: Can we generalize these facts to larger structures?

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- Assuming the Axiom of Choice there is a set reals without the perfect set property, but under ZFC every Σ^1_1 set of reals has the perfect set property.
- We can generalize this result by considering sets of reals in the structure $L(\mathbb{R})$.
- Assuming large cardinals, every set of reals in $L(\mathbb{R})$ has the perfect set property (Woodin).

the perfect set property

- In fact (assuming large cardinals), all classical regularity properties (Lebesgue measurability, property of Baire, etc.) are true for all sets of reals in $L(\mathbb{R})$.

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- AD is a fundamental regularity property in the sense that (for the most part)

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- There is in fact a fundamental regularity property called *the Axiom of Determinacy* (AD) which holds in $L(\mathbb{R})$.
- AD is a fundamental regularity property in the sense that (for the most part)

$$\forall(\text{regularity properties X})(AD \to X).$$

- Question: Can we generalize the above situation to larger structures? That is, we want to find a larger structure with similar regularity properties, and we want to find a 'fundamental regularity property'.

the strongest large cardinals

Theorem (Kunen)

(AC) There is no (non-trivial) elementary embedding

$$j : V \to V.$$

In fact for any λ there is no (non-trivial) elementary embedding

$$j : V_{\lambda+2} \to V_{\lambda+2}.$$

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Definition

1. I_1 is the statement: for some λ, there exists an elementary embedding
$$j : V_{\lambda+1} \to V_{\lambda+1}.$$
2. I_3 is the statement: for some λ, there exists an elementary embedding
$$j : V_\lambda \to V_\lambda.$$

the axiom I_0

Definition (Woodin)

I_0 is the statement: there exists a λ such that there is an elementary embedding

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

with $\operatorname{crit}(j) < \lambda$.

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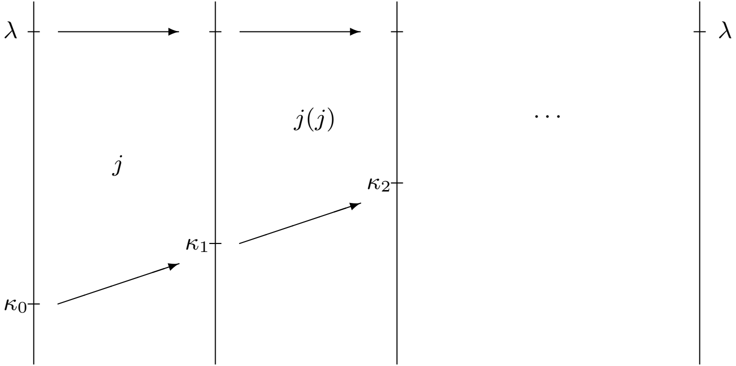
Woodin originally introduced I_0 in order to show that AD holds in $L(\mathbb{R})$ assuming large cardinals.

rank into rank embeddings

If $j : V_{\lambda+1} \to V_{\lambda+1}$ is elementary then λ is the sup of the critical sequence of j. That is, for $\kappa_0 = \text{crit}\,(j)$ and $\kappa_{i+1} = j(\kappa_i)$ for $i < \omega$, we have

$$\lambda = \sup_{i<\omega} \kappa_i.$$



relationship with $L(\mathbb{R})$

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- So $L(\mathbb{R}) = L(V_{\omega+1})$ and $L(V_{\lambda+1})$ are both structures of the form $L(V_{\alpha+1})$ for α a strong limit of cofinality ω.

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- So $L(\mathbb{R}) = L(V_{\omega+1})$ and $L(V_{\lambda+1})$ are both structures of the form $L(V_{\alpha+1})$ for α a strong limit of cofinality ω.
- Furthermore, assuming AD holds in $L(\mathbb{R})$, $L(\mathbb{R})$ does not satisfy the axiom of choice. And if I_0 holds at λ then $L(V_{\lambda+1})$ does not satisfy the axiom of choice either.

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- So $L(\mathbb{R}) = L(V_{\omega+1})$ and $L(V_{\lambda+1})$ are both structures of the form $L(V_{\alpha+1})$ for α a strong limit of cofinality ω.
- Furthermore, assuming AD holds in $L(\mathbb{R})$, $L(\mathbb{R})$ does not satisfy the axiom of choice. And if I_0 holds at λ then $L(V_{\lambda+1})$ does not satisfy the axiom of choice either.
- Do $L(\mathbb{R})$ and $L(V_{\lambda+1})$ have similar structural properties? For instance does $L(V_{\lambda+1})$ have similar combinatorial properties at λ^+ as ω_1 does in $L(\mathbb{R})$?

relationship with $L(\mathbb{R})$

Theorem

Assume AD holds in $L(\mathbb{R})$. Then $L(\mathbb{R})$ satisfies the following:

1. *ω_1 is measurable. In fact the club filter is an ultrafilter on ω_1 (Solovay).*
2. *Θ is a limit of measurable cardinals (Moschovakis).*

Definition

Let $\Theta = \Theta_\lambda = \sup\{\alpha | \text{(there exists a surjection of } V_{\lambda+1} \text{ onto } \alpha)^{L(V_{\lambda+1})}\}$.

Theorem (Woodin)

Assume I_0 holds at λ. Then the following hold in $L(V_{\lambda+1})$.

1. *λ^+ is measurable.*
2. *Θ is a limit of measurable cardinals.*

the club filter on λ^+

1. Woodin showed that in $L(V_{\lambda+1})$ the club filter restricted to some stationary set is an ultrafilter on λ^+. In fact, he showed that there is a partition $\langle T_\alpha \,|\, \alpha < \beta\rangle$ of λ^+ into stationary sets such that $\beta < \lambda$ and for all $\alpha < \beta$, the club filter restricted to T_α is an ultrafilter.

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2. Let $S_\alpha = \{\beta < \lambda^+ \,|\, \mathrm{cof}(\beta) = \alpha\}$. Question: Is the club filter restricted to S_α an ultrafilter?

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4. It is open whether or not the club filter restricted to S_ω is an ultrafilter in $L(V_{\lambda+1})$.

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4. It is open whether or not the club filter restricted to S_ω is an ultrafilter in $L(V_{\lambda+1})$.

Theorem (C.)

Assume I_0 at λ. Then there are no disjoint stationary subsets T_1, T_2 of S_ω (in V) such that $T_1, T_2 \in L(V_{\lambda+1})$.

partition relation on λ^+

Theorem (Woodin)

Suppose I_0 holds at λ. Then for all $\alpha < \beta < \omega_1$,

$$L_\lambda(H(\lambda^+)) \models \lambda^+ \to (\beta)^\alpha_\lambda.$$

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1. It is open whether or not for all $\alpha < \omega_1$,

$$\lambda^+ \to (\lambda^+)^\alpha_\lambda.$$

2. Since ω_1-DC holds in $L(V_{\lambda+1})$, we have that in $L(V_{\lambda+1})$

$$\lambda^+ \not\to (\lambda^+)^{\omega_1}.$$

   So it is not clear how to define a 'strong partition property' for $L(V_{\lambda+1})$.

perfect set property

Theorem (Davis)

Assume AD holds in $L(\mathbb{R})$. Then every set of reals in $L(\mathbb{R})$ has the perfect set property. That is if $X \subseteq \mathbb{R}$ and $X \in L(\mathbb{R})$ then either X is countable or X contains a perfect set and hence $|X| = 2^\omega$.

Theorem (C.)

Assume I_0 at λ. Then every subset $X \subseteq V_{\lambda+1}$ such that $X \in L(V_{\lambda+1})$ has the λ-splitting perfect set property. That is either $|X| \leq \lambda$ or X contains a λ-splitting perfect set and hence $|X| = 2^\lambda$.

analog of AD for $L(V_{\lambda+1})$

- The above results point to the possibility that I_0 for $L(V_{\lambda+1})$ is analogous to AD for $L(\mathbb{R})$.

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- There is a problem with this however:

Definition

For $X \subseteq V_{\lambda+1}$, let $I_0(X)$ be the statement that there exists an elementary embedding

$$j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$$

with $\operatorname{crit}(j) < \lambda$.

- We have

$$AD \to \text{the perfect set property}$$

but

$$I_0(X) \nrightarrow \text{the } \lambda\text{-splitting perfect set property.}$$

inverse limit reflection

- However there is a property called 'inverse limit reflection' (ILR) such that if I_0 holds at λ then $L(V_{\lambda+1})$ satisfies ILR. Furthermore

  $$\text{ILR} \rightarrow \text{the } \lambda\text{-splitting perfect set property.}$$

  So ILR is in this sense a better analog of AD for $L(V_{\lambda+1})$ than I_0.

reflecting I_3, I_1, and I_0

Theorem

1. *(I_1 reflects I_3) Suppose there is $V_{\lambda+1} \to V_{\lambda+1}$ an elementary embedding. Then there is a $\bar{\lambda} < \lambda$ and an elementary embedding $V_{\bar{\lambda}} \to V_{\bar{\lambda}}$ (Martin).*

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2. *(I_0 reflects I_1) Suppose there is $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ an elementary embedding with crit$(j) < \lambda$. Then there is a $\bar{\lambda} < \lambda$ and an elementary embedding $V_{\bar{\lambda}+1} \to V_{\bar{\lambda}+1}$ (Woodin).*

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3. *Assume there exists $j : L_{\lambda^+ +\omega+1}(V_{\lambda+1}) \to L_{\lambda^+ +\omega+1}(V_{\lambda+1})$ elementary. Then there exists a $\bar\lambda < \lambda$ such that there is an elementary embedding $k : L_{\bar\lambda^+}(V_{\bar\lambda+1}) \to L_{\bar\lambda^+}(V_{\bar\lambda+1})$ with crit$(k) < \bar\lambda$ (Laver).*

Laver used a technique called 'inverse limits' to get his reflection result.

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4. *($I_0^\#$ reflects I_0) Assume there exists an elementary embedding*

   $$j : L(V^\#_{\lambda+1}) \to L(V^\#_{\lambda+1})$$

   with crit$(j) < \lambda$. Then there exists a $\bar{\lambda} < \lambda$ and an elementary embedding

   $$k : L(V_{\bar{\lambda}+1}) \to L(V_{\bar{\lambda}+1})$$

   with crit$(k) < \bar{\lambda}$. (C.)

Laver used a technique called 'inverse limits' to get his reflection result.

definition of inverse limits

Definition (Laver)

An inverse limit $(J, \langle j_i | \, i < \omega \rangle)$ is a tuple such that the following hold:

1. For all $i < \omega$, $j_i : V_{\lambda+1} \to V_{\lambda+1}$ is elementary.
2. $\operatorname{crit}(j_0) < \operatorname{crit}(j_1) < \operatorname{crit}(j_2) < \cdots < \lambda$.
3. $\sup_{i<\omega} \operatorname{crit}(j_i) = \bar{\lambda} < \lambda$.
4. $J : V_{\bar{\lambda}+1} \to V_{\lambda+1}$ is defined by: for all $a \in V_{\bar{\lambda}}$,

$$J(a) = \lim_{i \to \omega} (j_0 \circ \cdots \circ j_i)(a) = (j_0 \circ j_1 \circ \cdots)(a).$$

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$$J(a) = \lim_{i \to \omega} (j_0 \circ \cdots \circ j_i)(a) = (j_0 \circ j_1 \circ \cdots)(a).$$

- If $(J, \langle j_i | \, i < \omega \rangle)$ is an inverse limit then we write

$$J = j_0 \circ j_1 \circ \cdots .$$

- We can rewrite an inverse limit as a direct limit as follows:

$$J = \cdots \circ j_0(j_1(j_2)) \circ j_0(j_1) \circ j_0.$$

picture of an inverse limit

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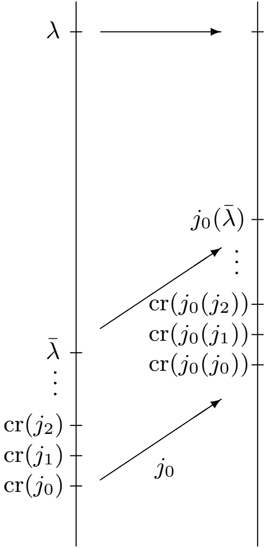

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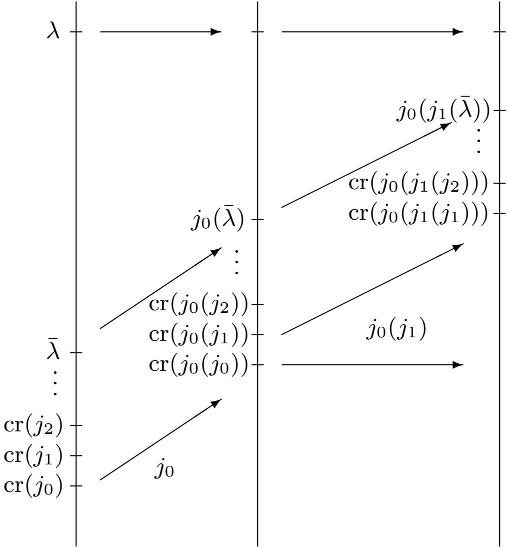

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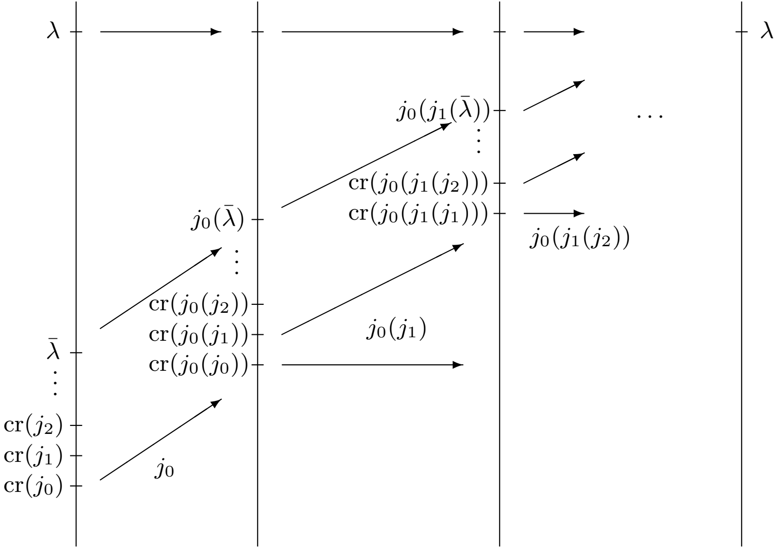

properties of inverse limits

- There are many theorems on inverse limits which take the basic form:

$$\text{property X for the embeddings } k_i \text{ for all } i < \omega$$
$$\Rightarrow \text{property } X \text{ for } K = k_0 \circ k_1 \circ \cdots$$

We say that property X *transfers* to inverse limits.

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  We say that property X *transfers* to inverse limits.

- For instance for (certain) inverse limits $K = k_0 \circ k_1 \circ \cdots$ we have for any $a \in V_{\lambda+1}$

$$\forall i < \omega(a \in \operatorname{rng} k_i) \to a \in \operatorname{rng} K.$$

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- For $j, k : V_{\lambda+1} \to V_{\lambda+1}$ elementary embeddings k is a *square root* of j if $k(k \upharpoonright V_\lambda) = j \upharpoonright V_\lambda$.

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$$\forall i < \omega (a \in \operatorname{rng} k_i) \rightarrow a \in \operatorname{rng} K.$$

- For $j, k : V_{\lambda+1} \rightarrow V_{\lambda+1}$ elementary embeddings k is a *square root* of j if $k(k \restriction V_\lambda) = j \restriction V_\lambda$.
- $K = k_0 \circ k_1 \circ \cdots$ is a *inverse limit root* of $J = j_0 \circ j_1 \circ \cdots$ if k_i is a square root of j_i for all large enough $i < \omega$.

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- For instance for (certain) inverse limits $K = k_0 \circ k_1 \circ \cdots$ we have for any $a \in V_{\lambda+1}$
$$\forall i < \omega (a \in \operatorname{rng} k_i) \to a \in \operatorname{rng} K.$$
- For $j, k : V_{\lambda+1} \to V_{\lambda+1}$ elementary embeddings k is a *square root* of j if $k(k \restriction V_\lambda) = j \restriction V_\lambda$.
- $K = k_0 \circ k_1 \circ \cdots$ is a *inverse limit root* of $J = j_0 \circ j_1 \circ \cdots$ if k_i is a square root of j_i for all large enough $i < \omega$.
- For E a set of inverse limits, $\mathrm{CL}(E)$ is the set of inverse limits $J = j_0 \circ j_1 \circ \cdots$ such that for all $n < \omega$ there is $K = k_0 \circ k_1 \circ \cdots \in E$ with $(k_0, \ldots, k_n) = (j_0, \ldots, j_n)$.

inverse limit reflection

Definition

Inverse limit reflection at α is the statement that there is a collection E of inverse limits satisfying the following.

1. E is closed under taking inverse limit roots in the sense that for all $J \in E$ and $x \in V_{\lambda+1}$, there is $K \in E$ an inverse limit root of J such that $x \in \operatorname{rng} K$.
2. The property 'extension to $L_\alpha(V_{\lambda+1})$' transfers to inverse limits on $\mathrm{CL}(E)$. In fact, there are unique $\bar{\alpha}$ and $\bar{\lambda}$ such that for all $J \in \mathrm{CL}(E)$, J extends to an elementary embedding

$$\hat{J} : L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to L_\alpha(V_{\lambda+1}).$$

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Theorem

Suppose I_0 holds at λ.

1. *Inverse limit reflection holds at λ^+ (Laver).*
2. *For all $\alpha < \Theta_\lambda$, inverse limit reflection holds at α (C.).*